Recall:

1. Bilinear forms $\longrightarrow$ Gram matrices

   Isometric classes $\qquad$ Congruent classes

2. Symmetric forms $\longrightarrow$ Symmetric matrices

3. Sylvester

$$A \rightsquigarrow P^T A P = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{bmatrix}$$

p positive index of inertia.

q negative index of inertia

Uniqueness uses positive definiteness.

Defn (positive definite) $V, \langle , \rangle$ symmetric form is positive definite iff

$\forall\, v \neq 0 \in V, \quad \langle v, v \rangle > 0.$

(or $\langle v, v \rangle \geq 0 \ \forall\, v \in V$ and equality holds iff $v = 0$)

Denote by $\langle , \rangle > 0$

Ex: $\mathbb{R}^n, \langle , \rangle$ standard

Prop: $V, \langle , \rangle$ positive definite iff
($\dim V < \infty$) $\dim V =$ positive index of inertia

In the proof of uniquess of signature.

positive index $= \max\{\dim W \mid W \subset V$ subspace, $\langle , \rangle|_W > 0\}$

This is a characterization of signature purely by isometry class of $(V, \langle , \rangle)$

Similarly, define negative definiteness.

Defn (negative definite)

V, $\langle , \rangle$ symmetric form.

$\langle , \rangle < 0$. iff $\forall v \neq 0 \in V$, $\langle v, v \rangle < 0$

Some related defn:

Defn (positive semidefinite)

$\langle , \rangle \geq 0$ iff $\forall v \in V$, $\langle v, v \rangle \geq 0$

Defn (negative semi-definite)

$\langle , \rangle \leq 0$ iff $\forall v \in V$, $\langle v, v \rangle \leq 0$

Defn: $(V, \langle , \rangle)$ with $\langle , \rangle > 0$

is called Euclidean space, or

inner product space

Prop: $\dim V = n$, inner product space, then

$(V, \langle , \rangle) \cong \mathbb{R}^n, \langle , \rangle_{\text{standard}}$

Ex: $P_{\leq n}(\mathbb{R}) = \{ f \in \mathbb{R}[x] \mid \deg f \leq n \}$

$\langle f, g \rangle = \int_0^1 fg \, dx$

$\langle , \rangle > 0$.

$P_{\leq n}(\mathbb{R})$ has basis $1, x, x^2 \ldots x^n$

Gram matrix under this basis $\neq I_{n+1}$

How to find more natural basis

Gram-Schmidt process.

$V, \langle\, , \rangle$ inner product space.

$B: v_1, \cdots v_n$ basis, try to find

basis $C: w_1, \cdots w_n$, s.t.

$$G_{\langle , \rangle, C} = I_n$$

or $\langle w_i, w_j \rangle = \delta_{ij}$

Defn (Orthonormal basis)
Such a basis is called orthonormal basis.

Idea: use the modification method in the proof of Sylvester theorem.

$v_1 \neq 0 \Rightarrow \langle v_1, v_1 \rangle \neq 0$

Define $w_1 = \dfrac{1}{\sqrt{\langle v_1, v_1 \rangle}} v_1$, then

$\langle w_1, w_1 \rangle = 1$

and $\operatorname{span}(w_1) = \operatorname{span}(v_1)$

$\tilde{w}_2 = v_2 - \langle v_2, w_1 \rangle w_1$. then

$\langle \tilde{w}_2, w_1 \rangle = 0$. and

$\operatorname{span}(w_1, \tilde{w}_2) = \operatorname{span}(v_1, v_2)$

so $\tilde{w}_2 \neq 0$,

Define $w_2 = \dfrac{1}{\sqrt{\langle \tilde{w}_2, \tilde{w}_2 \rangle}} \tilde{w}_2$

$\langle w_1, w_2 \rangle = 0$. $\langle w_2, w_2 \rangle = 1$

and $\operatorname{span}(w_1, w_2) = \operatorname{span}(v_1, v_2)$

Define $\tilde{W}_3 = V_3 - \langle V_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$

$\Rightarrow \langle \tilde{W}_3, w_1 \rangle = 0, \quad \langle \tilde{W}_3, w_2 \rangle = 0$

$\text{span}(\tilde{W}_3, w_1, w_2) = \text{span}(v_3, w_1, w_2)$

$\Rightarrow \tilde{w}_3 \neq 0 \qquad = \text{span}(v_3, v_1, v_2)$

Define $\quad w_3 = \dfrac{1}{\sqrt{\langle \tilde{w}_3, \tilde{w}_3 \rangle}} \tilde{w}_3$

$\langle w_3, w_1 \rangle = 0, \quad \langle w_3, w_2 \rangle = 0$

$\langle w_3, w_3 \rangle = 1$

$\text{span}(w_1, w_2, w_3) = \text{span}(v_1, v_2, v_3)$

Inductively define

$$\tilde{w}_i = v_i - \sum_{j=1}^{i-1} \langle v_i, w_j \rangle w_j$$

and $\quad w_i = \dfrac{1}{\sqrt{\langle \tilde{w}_i, \tilde{w}_i \rangle}} \tilde{w}_i$

then $w_1 \cdots w_n$ orthonormal and

$\mathrm{span}(w_1 \cdots w_i) = \mathrm{span}(v_1 \cdots v_i)$

In terms of transition matrix

$$(w_1 \cdots w_n) = (v_1 \cdots v_n) \cdot P$$

$$P = \begin{bmatrix} a_1 & & & * \\ 0 & a_2 & & \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \ddots \\ 0 & 0 & & a_n \end{bmatrix}$$

$\boxed{a_i > 0}$ $\qquad P$ upper triangular

because

flag structure is preserved

a sequence of subspaces.

and

$$(v_1 \cdots v_n) = (w_1 \cdots w_n) \cdot \underbrace{P^{-1}}$$

upper triangular
with positive digonal
elements.

Matrix version (QR decomposition)

Defn (orthogonal matrix)

$Q \in M_n(\mathbb{R})$ is called orthogonal matrix iff column vectors of Q form an orthonormal basis of $\mathbb{R}^n, \langle \, \rangle_{st}$

$\Leftrightarrow$ $Q^T Q = I_n$ (by $\langle x, y \rangle_{st} = x^T y$)

$\Leftrightarrow$ $QQ^T = I_n$ (by left inverse = right inverse)

$\Leftrightarrow$ row vectors of Q form an orthonormal basis

Thm (QR decomposition)

$\forall A \in GL(n, \mathbb{R})$, $\exists$ Q orthogonal matrix and R upper triangular matrix with positive diagonal entries, s.t.

$$A = Q \cdot R$$

Pf: $A = (v_1 \cdots v_n)$ form a basis of $\mathbb{R}^n$ iff A is invertible.

G-S process $\Rightarrow$

$$(v_1, \cdots v_n) = (w_1, \cdots w_n) \cdot R$$

$w_1, \cdots w_n$ orthonormal basis

so $\quad A = Q \cdot R$. $\quad \square$

uniqueness. $\quad$ If $\quad A = Q_1 R_1$

$$= Q_2 R_2$$

two QR decompositions.

$Q_1 = Q_2$?

$R_1 = R_2$? $\quad$ (Homework)